

NONLINEAR LIOUVILLE PROBLEMS IN A QUARTER PLANE

CHANG-LIN XIANG

ABSTRACT. We answer affirmatively the open problem proposed by Cabré and Tan in their paper "Positive solutions of nonlinear problems involving the square root of the Laplacian" (see Adv. Math. **224** (2010), no. 5, 2052-2093).

Keywords: Nonexistence; Monotonicity; The method of moving spheres

2010 Mathematics Subject Classification: 35B09; 35B53; 35J60

CONTENTS

1. Introduction and main result	1
2. Some preliminaries	3
3. Proof of Theorem 1.2	4
4. An extension	7
References	8

1. INTRODUCTION AND MAIN RESULT

In this paper, we consider positive solutions of the nonlinear boundary value problem

$$(1.1) \quad \begin{cases} \Delta u = 0, & \text{in } \mathbb{R}_{++}^{n+1}, \\ u(x, y) > 0, & \text{in } \mathbb{R}_{++}^{n+1}, \\ u(0, y) = 0 & \text{on } \{x_n = 0, y \geq 0\}, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \{x_n > 0, y = 0\}, \end{cases}$$

where $n \geq 1$, $1 \leq p < \infty$, $\mathbb{R}_{++}^{n+1} = \{(x_1, x_2, \dots, x_n, y) \in \mathbb{R}^{n+1} : x_n > 0, y > 0\}$ and ν is the unit out normal to \mathbb{R}_{++}^{n+1} at $\{x_n > 0, y = 0\}$.

Problem (1.1) was probably studied first by Cabré and Tan [4]. The motivation comes from the study of the Gidas-Spruck [11] type apriori estimates for solutions of the nonlinear nonlocal problem

$$(1.2) \quad \begin{cases} A_{1/2}u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $A_{1/2}$ is the square root of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$. For the precise definition of $A_{1/2}$, we refer the readers to Cabré and Tan [4]. Problem (1.1) appears as one of the

two limiting equations when applying the method of blow-up to solutions of Eq. (1.2); the other related limiting equation is given by

$$(1.3) \quad \begin{cases} \Delta u = 0, & \text{in } \mathbb{R}_+^{n+1}, \\ u(x, y) > 0, & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial \mathbb{R}_+^{n+1}. \end{cases}$$

It is well known that Eq. (1.3) has no weak solutions for all $p < (n+1)/(n-1)$ when $n \geq 2$ (see e.g. [12, 16, 17, 19]). For related Liouville type problems in the whole space \mathbb{R}^{n+1} , we refer to e.g. Caffarelli et al. [2], Chen, Li and Ou [6, 8] and Y.Y. Li [14].

By the regularity theory developed in Cabré and Tan [4], solutions of Eq. (1.1) in the weak sense are shown to be classical in the sense that, any weak solution of Eq. (1.1) belongs to $C^2(\mathbb{R}_{++}^{n+1}) \cap C^1(\overline{\mathbb{R}_{++}^{n+1}})$. Thus, we restrict our attention to classical solutions of Eq. (1.1). As one of their main results, Cabré and Tan [4] obtained the following result (see [4, Theorem 1.5]).

Theorem 1.1. *Let $n \geq 2$ and $1 < p \leq (n+1)/(n-1)$. Then, there exists no bounded classical solution to Eq. (1.1).*

Equivalently, there exists no bounded solution of equation

$$\begin{cases} A_{1/2}u = u^p & \text{in } \mathbb{R}_+^n, \\ u > 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 & \text{on } \partial \mathbb{R}_+^n, \end{cases}$$

where $A_{1/2}$ is the square root of the Laplacian in $\mathbb{R}_+^n = \{x_n > 0\}$ with zero Dirichlet boundary conditions on $\partial \mathbb{R}_+^n$.

We briefly review the approach of Cabré and Tan [4] in below for later use. Let $n \geq 2$ and $1 < p \leq (n+1)/(n-1)$. Suppose that u is a classical solution to Eq. (1.1). First Cabré and Tan [4] derived the symmetry of u with respect to x_i , $1 \leq i \leq n-1$, by combining the Kelvin transform and the method of moving planes. Since Eq. (1.1) is translation invariant with respect to x_i , $1 \leq i \leq n-1$, it follows that u depends only on x_n and y (see [4, Proposition 6.3]). Hence, Eq. (1.1) is reduced to the following problem in the two dimensional quarter plane

$$(1.4) \quad \begin{cases} \Delta u = 0, & \text{in } \mathbb{R}_{++}^2 = \{x > 0, y > 0\}, \\ u > 0, & \text{in } \mathbb{R}_{++}^2, \\ u = 0 & \text{on } \{x = 0, y \geq 0\}, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \{x > 0, y = 0\}. \end{cases}$$

Then they proved that Eq. (1.4) has no bounded classical solution by applying a Hamiltonian identity for the half-Laplacian found by Cabré and Solà-Morales [3]. In this way, Theorem 1.1 is proved.

Some remarks are in order. First, we remark that to reduce Eq. (1.1) to Eq. (1.4), the boundedness assumption of the solution is not needed. Thus the boundedness assumption is only used when deriving the nonexistence of solutions of Eq. (1.4). Next, we remark that,

under the boundedness assumption of the solution, Cabré and Tan [4] derived nonexistence results for equations of type (1.4) under far more general boundary conditions (see Cabré and Tan [4, Proposition 6.4]). However, they pointed out that Theorem 1.1 is open without the assumption of boundedness of the solution.

In this paper, we remove their boundedness assumption. The following theorem is our main result.

Theorem 1.2. *Let $n \geq 1$. Assume that $1 \leq p \leq (n+1)/(n-1)$ for $n \geq 2$ and $1 \leq p < \infty$ for $n = 1$. Then, there exists no classical solution to Eq. (1.1).*

We prove Theorem 1.2 in Section 3. The idea is as follows. First note that by the symmetry result of Cabré and Tan [4, Proposition 6.3], Eq. (1.1) is reduced to Eq. (1.4). Then, as a key gradient, we show that any positive solution of Eq. (1.4) is monotone increasing in the x -direction. This idea is inspired by the work of Li and Lin [15], where a nonlinear elliptic PDE with two Sobolev-Hardy critical exponents are considered. Finally, combining the monotonicity result together with the very general result of Cabré and Tan [4, Proposition 6.2] (see Proposition 2.2 below), we obtain Theorem 1.2. To complete the proof of Theorem 1.2, we will give some necessary results in the next Section 2. In the last section, we give an extension of Theorem 1.2, which can be seen as an analogue of Cabré and Tan [4, Proposition 6.4].

Our notations are standard. $B_R(x)$ is the open ball in \mathbb{R}^N centered at x with radius $R > 0$. Whenever $E \subset \mathbb{R}^N$ is a Lebesgue measurable set, we denote by $|E|$ the N -dimensional Lebesgue measure of set E . Let Ω be an arbitrary domain in \mathbb{R}^N . For any $1 \leq s \leq \infty$, $L^s(\Omega)$ is the Banach space of Lebesgue measurable functions u such that the norm

$$\|u\|_{s,\Omega} = \begin{cases} \left(\int_{\Omega} |u|^s \right)^{\frac{1}{s}} & \text{if } 1 \leq s < \infty \\ \text{esssup}_{\Omega} |u| & \text{if } s = \infty \end{cases}$$

is finite. A function u belongs to the Sobolev space $W^{1,s}(\Omega)$ if $u \in L^s(\Omega)$ and its first order weak partial derivatives also belong to $L^s(\Omega)$. For the properties of the Sobolev functions, we refer to the monograph [23].

2. SOME PRELIMINARIES

In this section we collect some useful results for later use. The first one concerns with Sobolev-Poincaré type inequalities in planar domains, which will be used in the proof of Lemma 3.1.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}_+^2$ be a bounded Lipschitz domain with a partial boundary $\Gamma \subset \partial\mathbb{R}_+^2$ (Γ could be an empty set). Then for any number q , $1 \leq q < \infty$, there exists a constant C_q , depending only on q , such that the following inequality holds*

$$\|u\|_{q,\Omega} \leq C_q |\Omega|^{\frac{1}{q}} \|\nabla u\|_{2,\Omega}$$

for all functions $u \in W^{1,2}(\Omega) \cap C(\Omega \cup \Gamma)$ with $u = 0$ on $\partial\Omega \cap \mathbb{R}_+^2$.

Proof. This lemma may be well known to specialist. We give a sketch of proof for the reader's convenience.

First consider the case $\Gamma = \emptyset$. In this case, Lemma 2.1 is a direct consequence of the Trudinger-Moser inequality (see [18, 20, 22])

$$\sup_{\|\nabla u\|_{2,\Omega}=1} \int_{\Omega} e^{\alpha|u|^2} \leq C_{\alpha}|\Omega|,$$

where $\alpha \leq 4\pi$ and $C_{\alpha} > 0$ is a constant depending only on α . Take $\alpha = 1$. We obtain that

$$\|u\|_{2k,\Omega} \leq C_k |\Omega|^{\frac{1}{2k}} \|\nabla u\|_{2,\Omega}$$

for all $k \in \{1, 2, \dots\}$. Now, Lemma 2.1 follows easily from above and Hölder's inequality in the case $\Gamma = \emptyset$.

In the general case when $\Gamma \neq \emptyset$, it suffices to consider the even extension

$$\tilde{u}(x, y) = \begin{cases} u(x, y) & \text{for } y \geq 0 \\ u(x, -y) & \text{for } y < 0 \end{cases}$$

for $u \in W^{1,2}(\Omega) \cap C(\Omega \cup \Gamma)$ with $u = 0$ on $\partial\Omega \cap \mathbb{R}_+^2$. Then this case is reduced to the previous one. The proof of Lemma 2.1 is finished. \square

The next very general result is Proposition 6.2 of Cabré and Tan [4] (see also Chipot et al. [9]), which will be used in the proof of Theorem 1.2.

Proposition 2.2. *Suppose that v weakly solves*

$$\begin{cases} -\Delta v \geq 0 & \text{in } \mathbb{R}_+^2, \\ v \geq 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial v}{\partial \nu} \geq 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Then v is a constant.

3. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. As already reviewed the approach of Cabré and Tan [4] in the introduction part, to prove Theorem 1.2, we only need to prove that Eq. (1.4) has no classical solution. We use the following lemma as a key gradient of the proof.

Lemma 3.1. *Suppose that u is a classical positive solution to Eq. (1.4). Then $u_x(x, y) > 0$ for all $(x, y) \in \overline{\mathbb{R}_{++}^2}$.*

Before giving a proof of Lemma 3.1, we will apply Lemma 3.1 to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that u is a positive solution to Eq. (1.4). Define the odd extension $\bar{u} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ of u by

$$\bar{u}(x, y) = \begin{cases} u(x, y) & \text{if } x \geq 0 \\ -u(-x, y) & \text{if } x \leq 0. \end{cases}$$

Since $u(0, y) \equiv 0$ for $y \geq 0$, it is elementary to find that \bar{u} solves equation

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial \bar{u}}{\partial \nu} = |\bar{u}|^{p-1} \bar{u} & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Furthermore, we deduce from above equation that \bar{u}_x satisfies

$$(3.1) \quad \begin{cases} \Delta \bar{u}_x = 0 & \text{in } \mathbb{R}_+^2, \\ \bar{u}_x(x, y) = u_x(|x|, y) > 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial \bar{u}_x}{\partial \nu} = p|\bar{u}|^{p-1}\bar{u}_x \geq 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Applying Proposition 2.2 to Eq. (3.1) gives that $\bar{u}_x \equiv C$ in \mathbb{R}_+^2 for some constant $C > 0$. Since $\bar{u}(0, y) \equiv 0$ for $y \geq 0$, we derive that $\bar{u}(x, y) = Cx$ for all $(x, y) \in \mathbb{R}_+^2$. But then, it follows that $\partial_\nu u \equiv 0 \neq u^p$ on $\{x > 0, y = 0\}$. We reach a contradiction. The proof of Theorem 1.2 is complete. \square

Now we prove Lemma 3.1. We will employ the method of moving spheres (see Li, Zhang and Zhu [14, 16, 17]), a variant of the method of moving planes invented by the Soviet mathematician Alexanderov in the early 1950s, and later further developed by Serrin [21], Gidas et al. [10], Caffarelli et al. [2], Li [13], Chen and Li [6, 7], Chang and Yang [5], Chen et al. [8] and many others. We also make use of the idea of narrow domains from Berestycki and Nirenberg [1].

Proof of Lemma 3.1. First we introduce some notations for convenience. Denote the point in the plane by $z = (x, y) \in \mathbb{R}^2$. Let $\lambda, R \in (0, \infty)$, $\lambda > R$, be arbitrary positive constants and write $z_R = (-R, 0)$. For any positive solution u of Eq. (1.4), define the function $u_{R,\lambda} : \Omega_{R,\lambda} \rightarrow [0, \infty)$ by

$$u_{R,\lambda}(z) = u\left(z_R + \frac{\lambda^2(z - z_R)}{|z - z_R|^2}\right) \quad \text{for } z \in \Omega_{R,\lambda},$$

where $\Omega_{R,\lambda}$ is the bounded domain given by

$$\Omega_{R,\lambda} = B_\lambda(z_R) \cap \mathbb{R}_{++}^2.$$

Since u solves Eq. (1.4), a direct calculation shows that $u_{R,\lambda}$ satisfies

$$(3.2) \quad \begin{cases} \Delta u_{R,\lambda} = 0 & \text{in } \Omega_{R,\lambda}, \\ u_{R,\lambda} > 0 & \text{in } \Omega_{R,\lambda}, \\ u_{R,\lambda} = u & \text{on } \partial\Omega_{R,\lambda} \cap \mathbb{R}_{++}^2, \\ \frac{\partial u_{R,\lambda}}{\partial \nu} = \left(\frac{\lambda}{|z - z_R|}\right)^2 u_{R,\lambda}^p & \text{on } \partial\Omega_{R,\lambda} \cap \{x > 0, y = 0\}. \end{cases}$$

Our aim is to show that

$$(3.3) \quad u(z) < u_{R,\lambda}(z) \quad \text{in } \Omega_{R,\lambda}$$

for all $\lambda, R \in (0, \infty)$ with $\lambda > R$.

Let $R > 0$ be fixed. First we show that (3.3) holds when $\lambda - R > 0$ is sufficiently small. To this end, set $w_\lambda(z) = u(z) - u_{R,\lambda}(z)$ for $z \in \Omega_{R,\lambda}$. We have that

$$(3.4) \quad \begin{cases} \Delta w_\lambda = 0 & \text{in } \Omega_{R,\lambda}, \\ w_\lambda = 0 & \text{on } \partial\Omega_{R,\lambda} \cap \mathbb{R}_{++}^2, \\ w_\lambda < 0 & \text{on } \partial\Omega_{R,\lambda} \cap \{x = 0, y > 0\}, \\ \frac{\partial w_\lambda}{\partial \nu} = u^p - \left(\frac{\lambda}{|z - z_R|}\right)^2 u_{R,\lambda}^p & \text{on } \partial\Omega_{R,\lambda} \cap \{x > 0, y = 0\}. \end{cases}$$

Multiply Eq. (3.4) by $w_\lambda^+ \equiv \max(w_\lambda, 0)$ and integrate by parts. We deduce that

$$\int_{\Omega_{R,\lambda}} |\nabla w_\lambda^+|^2 = \int_{\partial\Omega_{R,\lambda} \cap \{x>0, y=0\}} w_\lambda^+ \left(u^p - \left(\frac{\lambda}{|z - z_R|} \right)^2 u_{R,\lambda}^p \right).$$

Denote

$$A_\lambda = \{z \in \partial\Omega_{R,\lambda} \cap \{x > 0, y = 0\} : w_\lambda(z) > 0\}.$$

Since $\lambda > |z - z_R|$ on $\partial\Omega_{R,\lambda} \cap \{x > 0, y = 0\}$ and $p \geq 1$, we have that

$$\int_{\partial\Omega_{R,\lambda} \cap \{x>0, y=0\}} w_\lambda^+ \left(u^p - \left(\frac{\lambda}{|z - z_R|} \right)^2 u_{R,\lambda}^p \right) \leq \int_{A_\lambda} p u^{p-1} (w_\lambda^+)^2.$$

By the local boundedness of $u(x, 0)$ for $x > 0$, we have that

$$\int_{A_\lambda} p u^{p-1} (w_\lambda^+)^2 \leq p \sup_{x \in A_\lambda} u^{p-1}(x, 0) \int_{A_\lambda} (w_\lambda^+)^2.$$

Hence combining above estimates together with Hölder's inequality gives that

$$\int_{\Omega_{R,\lambda}} |\nabla w_\lambda^+|^2 \leq p \left(\sup_{x \in A_\lambda} u^{p-1}(x, 0) \right) |A_\lambda|^{1-\frac{2}{q}} \left(\int_{\Omega_{R,\lambda}} (w_\lambda^+)^q \right)^{\frac{2}{q}},$$

where $2 < q < \infty$ is a fixed number. Note that $w_\lambda^+ = 0$ on $\partial\Omega_{R,\lambda} \cap \{y > 0\}$. Applying Lemma 2.1 with $\Omega = \Omega_{R,\lambda}$, we deduce that

$$(3.5) \quad \int_{\Omega_{R,\lambda}} |\nabla w_\lambda^+|^2 \leq C_{p,q} \left(\sup_{x \in A_\lambda} u^{p-1}(x, 0) \right) |\Omega_{R,\lambda}|^{\frac{2}{q}} |A_\lambda|^{1-\frac{2}{q}} \int_{\Omega_{R,\lambda}} |\nabla w_\lambda^+|^2,$$

where $C_{p,q} > 0$ is a constant depending only on p and q .

Note that $|A_\lambda| \leq \lambda - R$. Thus, it is easy to infer from inequality (3.5) that (3.3) holds when $\lambda - R > 0$ is sufficiently small.

Next we show that for any fixed $R > 0$, (3.3) holds for all $\lambda \in (R, \infty)$. To this end, set

$$\bar{\lambda}(R) = \{\mu \in (R, \infty) : (3.3) \text{ holds for all } R < \lambda < \mu.\}$$

We claim that $\bar{\lambda}(R) = \infty$. Argue by contradiction. Suppose that $\bar{\lambda}(R) < \infty$ holds. Then by continuity, we have that $u \leq u_{R, \bar{\lambda}(R)}$ in $\Omega_{R, \bar{\lambda}(R)}$. Since $u < u_{R, \bar{\lambda}(R)}$ on $\partial\Omega_{R, \bar{\lambda}(R)} \cap \{x = 0, y > 0\}$, we deduce that $u < u_{R, \bar{\lambda}(R)}$ in $\Omega_{R, \bar{\lambda}(R)}$ by the strong maximum principle. Therefore we infer that

$$|A_\lambda| \rightarrow 0 \quad \text{as } \lambda \downarrow \bar{\lambda}(R).$$

Thus, there exists a sufficiently small number $\delta > 0$, such that

$$C_{p,q} \left(\sup_{0 < x < \bar{\lambda}(R) + \delta} u^{p-1}(x, 0) \right) |\Omega_{R, \bar{\lambda}(R) + \delta}|^{\frac{2}{q}} |A_\lambda|^{1-\frac{2}{q}} < \frac{1}{2}$$

for all $\lambda \in (\bar{\lambda}(R), \bar{\lambda}(R) + \delta)$. Then combining above estimate together with inequality (3.5) yields that $u \leq u_{R,\lambda}$ in $\Omega_{R,\lambda}$ for all $\lambda \in (\bar{\lambda}(R), \bar{\lambda}(R) + \delta)$. This is against the choice of $\bar{\lambda}(R)$. Hence we conclude that $\bar{\lambda}(R) = \infty$. In this way, we show that for any fixed $R > 0$, (3.3) holds for all $\lambda \in (R, \infty)$.

Now we can finish the proof of Lemma 3.1. Let (x_1, y_0) and (x_2, y_0) , $0 < x_1 < x_2$, be two arbitrary points in \mathbb{R}_{++}^2 . Then for all $R > 0$ sufficiently large, we have $(x_1, y) \in B_{R+a}(z_R) \cap \mathbb{R}_{++}^2$, where $a = (x_1 + x_2)/2$. Then applying (3.3) with $\lambda = R + a$ gives that

$$u(x_1, y_0) < u_{R,R+a}(x_1, y_0)$$

for all $R > 0$ sufficiently large. Letting $R \rightarrow \infty$ in the above inequality yields that

$$u(x_1, y_0) \leq u(2a - x_1, y_0) = u(x_2, y_0).$$

This shows that u is monotone increasing in the x -direction, that is, $u_x \geq 0$ in \mathbb{R}_{++}^2 .

To derive the strict inequality in Lemma 3.1, we note that u_x is also a harmonic function in \mathbb{R}_+^2 and $\partial_\nu u_x = pu^{p-1}u_x \geq 0$ on $\{x > 0, y = 0\}$. Hence it follows from the strong maximum principle that $u_x > 0$ in \mathbb{R}_{++}^2 , and from the Hopf lemma that $u_x(0, y) > 0$ on $\partial\mathbb{R}_{++}^2$. The proof of Lemma 3.1 is complete. \square

4. AN EXTENSION

Recall that we mentioned the quite general nonexistence result of Cabré and Tan [4, Proposition 6.4] in the introduction part. It states as follows.

Proposition 4.1. *Assume that f is a $C^{1,\alpha}$ function for some $\alpha \in (0, 1)$, such that $f > 0$ in $(0, \infty)$ and $f(0) = 0$. Let C be a positive constant. Then there is no bounded solution of the problem*

$$(4.1) \quad \begin{cases} \Delta u = 0, & \text{in } \mathbb{R}_{++}^2, \\ 0 < u(x, y) \leq C, & \text{in } \mathbb{R}_{++}^2, \\ u(0, y) = 0 & \text{on } \{x = 0, y \geq 0\}, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \{x > 0, y = 0\}. \end{cases}$$

In this section, we give an extension of Theorem 1.2 in the case $n = 1$, which can be seen as an analogue of Proposition 4.1.

Theorem 4.2. *Assume that $f : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing C^1 function with $f(0) = 0$, and that u is a nonnegative classical solution to the problem*

$$(4.2) \quad \begin{cases} \Delta u = 0, & \text{in } \mathbb{R}_{++}^2, \\ u(x, y) \geq 0, & \text{in } \mathbb{R}_{++}^2, \\ u(0, y) = 0 & \text{on } \{x = 0, y \geq 0\}, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \{x > 0, y = 0\}. \end{cases}$$

Then there exists a constant $C \geq 0$ such that

$$u(x, y) = Cx \quad \text{for } (x, y) \in \mathbb{R}_{++}^2.$$

Proof. We only give a sketch of the proof. Let u be a positive solution to Eq. (4.1).

First we show that u is nondecreasing in the x -direction. Define $\Omega_{R,\lambda}$ and $u_{R,\lambda} : \Omega_{R,\lambda} \rightarrow [0, \infty)$ as in the proof of Lemma 3.1. It is elementary to derive that

$$\begin{cases} \Delta u_{R,\lambda} = 0 & \text{in } \Omega_{R,\lambda}, \\ u_{R,\lambda} \geq 0 & \text{in } \Omega_{R,\lambda}, \\ u_{R,\lambda} = u & \text{on } \partial\Omega_{R,\lambda} \cap \mathbb{R}_{++}^2, \\ \frac{\partial u_{R,\lambda}}{\partial \nu} = \left(\frac{\lambda}{|z-z_R|}\right)^2 f(u_{R,\lambda}) & \text{on } \partial\Omega_{R,\lambda} \cap \{x > 0, y = 0\}. \end{cases}$$

Then set $w_\lambda = u - u_{R,\lambda}$ in $\Omega_{R,\lambda}$. Since f is nondecreasing and continuously differentiable, we deduce that

$$\int_{\Omega_{R,\lambda}} |\nabla w_\lambda^+|^2 \leq C_{p,q} \left(\sup_{0 \leq t \leq \|u\|_{\infty, A_\lambda}} f'(t) \right) |\Omega_{R,\lambda}|^{\frac{2}{q}} |A_\lambda|^{1-\frac{2}{q}} \int_{\Omega_{R,\lambda}} |\nabla w_\lambda^+|^2,$$

where A_λ is defined as in the proof of Lemma 3.1. Above inequality is a counterpart of (3.5). Thus we can conclude as in the proof of Lemma 3.1 that u is nondecreasing in the x -direction.

Next, consider the odd extension \bar{u} of u with respect to $\{x = 0, y > 0\}$. We deduce that \bar{u}_x satisfies

$$\begin{cases} \Delta \bar{u}_x = 0 & \text{in } \mathbb{R}_+^2, \\ \bar{u}_x(x, y) = u_x(|x|, y) \geq 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial \bar{u}_x}{\partial \nu} = \bar{f}'(\bar{u}) \bar{u}_x \geq 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

where \bar{f} is the odd extension of f , that is, $\bar{f}(t) = f(t)$ for $t \geq 0$ and $\bar{f}(t) = -f(-t)$ for $t < 0$. Now Theorem 4.2 follows from Proposition 2.2 easily. \square

In the spirit of Theorem 1.1, we have the following application of Theorem 4.2.

Corollary 4.3. *Assume that $f : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing C^1 function with $f(0) = 0$. Then, there exists no bounded solution to the problem*

$$\begin{cases} A_{1/2} u = f(u) & \text{in } \mathbb{R}_+ = \{x > 0\}, \\ u > 0 & \text{in } \mathbb{R}_+, \\ u(0) = 0, \end{cases}$$

where $A_{1/2}$ is the square root of the Laplacian in $(0, \infty)$ with zero Dirichlet boundary conditions at $x = 0$.

Acknowledgment. The author is financially supported by the Academy of Finland, project 259224.

REFERENCES

- [1] H. BERESTYCKI AND L. NIRENBERG, *Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations*. J. Geom. Phys. **5** (1988), no. 2, 237-275.
- [2] L. CAFFARELLI, B. GIDAS AND J. SPRUCK, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*. Comm. Pure Appl. Math. **42** (1989), 271-297.
- [3] X. CABRÉ AND SOLÀ-MORALES, *Layer solutions in a half space for boundary reactions*. Comm. Pure Appl. Math. **58** (2005), 1678-1732.

- [4] X. CABRÉ AND J. TAN, *Positive solutions of nonlinear problems involving the square root of the Laplacian*. Adv. Math. **224** (2010), no. 5, 2052-2093.
- [5] S.-Y. A. CHANG AND PAUL C. YANG, *On uniqueness of solutions of n th order differential equations in conformal geometry*. Math. Res. Lett. **4**, **1** (1997), 91-102.
- [6] W.-X. CHEN AND C. LI, *Classification of solutions of some nonlinear elliptic equations*. Duke Math. J. **63** (3) (1991), 615-622.
- [7] W.-X. CHEN AND C. LI, *A priori estimates for prescribing scalar curvature equations*. Ann. of Math. (2) **145**, 3 (1997), 547-564.
- [8] W. CHEN, C. LI AND B. OU, *Classification of solutions for an integral equation*. Comm. Pure Appl. Math. **59** (2006), no. 3, 330-343.
- [9] M. CHIPOT, M. CHLEBÍK, M. FILA AND I. SHAFRIR, *Existence of positive solutions of a semilinear elliptic equation in \mathbb{R}_+^n with a nonlinear boundary condition*. J. Math. Anal. Appl. **223** (1998), 429-471.
- [10] B. GIDAS, W.M. NI AND L. NIRENBERG, *Symmetry and related properties via the maximum principle*. Comm. Math. Phys. **68** (1979), no. 3, 209-243.
- [11] B. GIDAS AND J. SPRUCK, *A priori bounds for positive solutions of nonlinear elliptic equations*. Comm. Partial Differential Equations **6** (1981), no. 8, 883-901.
- [12] B. HU, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*. Differential Integral Equations **7** (1994), no. 2, 301-313.
- [13] C. LI, *Local asymptotic symmetry of singular solutions to nonlinear elliptic equations*. Invent. Math. **123** (1996), no. 2, 221-231.
- [14] Y.Y. LI, *Remark on some conformally invariant integral equations: the method of moving spheres*. J. Eur. Math. Soc. (JEMS) **6** (2004), no. 2, 153-180.
- [15] Y.Y. LI AND C.-S. LIN, *A nonlinear elliptic PDE and two Sobolev-Hardy critical exponents*. Arch. Ration. Mech. Anal. **203** (2012), no. 3, 943-968.
- [16] Y.Y. LI AND L. ZHANG, *Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations*. J. Anal. Math. **90** (2003), 27-87.
- [17] Y.Y. LI AND M. ZHU, *Uniqueness theorems through the method of moving spheres*. Duke Math. J. **80** (1995), no. 2, 383-417.
- [18] J. MOSER, *A sharp form of an inequality by N. Trudinger*. Ind. Univ. Math. J. **30** (1967), 473-484.
- [19] B. OU, *Positive harmonic functions on the upper half space satisfying a nonlinear boundary condition*. Differential Integral Equations **9** (1996), no. 5, 1157-1164.
- [20] S.I. POHOZÁEV, *The Sobolev embedding in the case $pl = n$* . Proc. Tech. Sci. Conf. on Adv. Sci. Research 1964-1965, Mathematics Section, Moskov. Energet. Inst. Moscow, 1965, pp. 158-170.
- [21] J. SERRIN, *A symmetry problem in potential theory*. Arch. Rational Mech. Anal. **43** (1971), 304-318.
- [22] N.S. TRUDINGER, *On imbeddings into Orlicz spaces and some applications*. J. Math. Mech. **75** (1980), 59-77.
- [23] W.P. ZIEMER, *Weakly differentiable functions*. Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

UNIVERSITY OF JYVASKYLA, DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. Box 35, FI-40014 UNIVERSITY OF JYVASKYLA, FINLAND.

E-mail address: Xiang_math@126.com